



Covering Problems for Functions n -Fold Symmetric and Convex in the Direction of the Real Axis II

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Abstract Let \mathcal{F} denote the class of all functions univalent in the unit disk $\Delta \equiv \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and convex in the direction of the real axis. The paper deals with the subclass $\mathcal{F}^{(n)}$ of these functions f which satisfy the property $f(\varepsilon z) = \varepsilon f(z)$ for all $z \in \Delta$, where $\varepsilon = e^{2\pi i/n}$. The functions of this subclass are called n -fold symmetric. For $\mathcal{F}^{(n)}$, where n is odd positive integer, the following sets, $\bigcap_{f \in \mathcal{F}^{(n)}} f(\Delta)$ —the Koebe set and $\bigcup_{f \in \mathcal{F}^{(n)}} f(\Delta)$ —the covering set, are discussed. As corollaries, we derive the Koebe and the covering constants for $\mathcal{F}^{(n)}$.

Keywords Covering domain · Koebe domain · Convexity in one direction · n -fold symmetry

Mathematics Subject Classification Primary 30C45 · Secondary 30C80

1 Introduction

Let \mathcal{F} denote the class of all functions f which are univalent in $\Delta \equiv \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, convex in the direction of the real axis and normalized by $f(0) = f'(0) - 1 = 0$. Recall that an analytic function f is said to be convex in the direction of the real axis if the intersection of $f(\Delta)$ with each horizontal line is either a connected set or empty.

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For a given subclass A of \mathcal{F} , the following sets: $\bigcap_{f \in A} f(\Delta)$ and $\bigcup_{f \in A} f(\Delta)$ are called the Koebe set for A and the covering set for A . We denote them by K_A and L_A , respectively. The radius of the largest disk with center at the origin contained in K_A is called the Koebe constant for A . Analogously, the radius of the smallest disk with center at the origin that contains L_A is called the covering constant for A .

In the class \mathcal{F} , we consider functions which satisfy the property of n -fold symmetry:

$$f(\varepsilon z) = \varepsilon f(z) \quad \text{for all } z \in \Delta,$$

where $\varepsilon = e^{2\pi i/n}$. The subclass of \mathcal{F} consisting of n -fold symmetric functions is denoted by $\mathcal{F}^{(n)}$. By the definition, for every $f \in \mathcal{F}^{(n)}$ a set $f(\Delta)$ is n -fold symmetric, which means that $f(\Delta) = \varepsilon f(\Delta)$. In other words, $f(\Delta)$ may be obtained as the union of rotations about a multiple of $2\pi/n$ from a set $f(\Delta) \cap \{w : \arg w \in [0, 2\pi/n]\}$. From this reason, the following notation is useful:

$$\Lambda_0 = \left\{ w : \arg w \in \left[0, \frac{2\pi}{n} \right] \right\}, \quad \Lambda_j = \varepsilon^j \Lambda_0, \quad j = 1, 2, \dots, n-1$$

and

$$\Lambda^* = \left\{ w : \arg w \in \left[\frac{\pi}{2} - \frac{\pi}{n}, \frac{\pi}{2} \right] \right\}.$$

The main aim of the paper is to find the Koebe set and the covering set for the class $\mathcal{F}^{(n)}$ when n is an odd positive integer. Similar problems in related classes were discussed, for instance, in [1, 2, 5] and in the papers of the authors [3, 4].

At the beginning, let us consider the general properties of the Koebe sets and the covering sets for $\mathcal{F}^{(n)}$.

In [4], we proved that

Theorem 1 *The sets $K_{\mathcal{F}^{(n)}}$ and $L_{\mathcal{F}^{(n)}}$, for $n \in \mathbb{N}$, are symmetric with respect to both axes of the coordinate system.*

Theorem 2 *The sets $K_{\mathcal{F}^{(n)}}$ and $L_{\mathcal{F}^{(n)}}$, for $n \in \mathbb{N}$, are n -fold symmetric.*

To prove both the theorems, it is enough to consider functions

$$g(z) = \overline{f(\bar{z})} \tag{1}$$

and

$$h(z) = -f(-z). \tag{2}$$

Obviously,

$$f \in \mathcal{F}^{(n)} \Leftrightarrow g, h \in \mathcal{F}^{(n)}. \tag{3}$$

Moreover, if $D = f(\Delta)$ then $g(\Delta) = \overline{D}$, $h(\Delta) = -D$.

Taking (3) into account, it is clear that the coordinate axes are the lines of symmetry for both the sets $K_{\mathcal{F}^{(n)}}$ and $L_{\mathcal{F}^{(n)}}$ for all $n \in \mathbb{N}$. Furthermore,

Lemma 1 *Each straight line $\varepsilon^{j/4} \cdot \{\zeta = t, t \in R\}$, $j = 0, 1, \dots, 4n - 1$ is the line of symmetry of $K_{\mathcal{F}(n)}$ and $L_{\mathcal{F}(n)}$ for every positive odd integer n .*

Proof Let n be a positive odd integer and let D be one of the two sets: $K_{\mathcal{F}(n)}$ or $L_{\mathcal{F}(n)}$.

Since D is symmetric with respect to the real axis and the positive real half-axis contains one side of the sector Λ_0 , each rotation of the real axis about a multiple of $2\pi/n$ is the line of symmetry of D . Because of the equality

$$\{\zeta = t, t \in R\} \cdot \varepsilon^{1/2} = \{\zeta = t, t \in R\} \cdot \varepsilon^{(n+1)/2},$$

our claim is true for all even j , $j = 0, 1, \dots, 4n - 1$.

Let $n = 4k + 1$, $k \geq 1$. The bisector of $\Lambda_{(n-1)/4}$ divides this sector into two subsectors: $\{w : \arg w \in [\pi/2 - \pi/2n, \pi/2 + \pi/2n]\}$ and $\{w : \arg w \in [\pi/2 + \pi/2n, \pi/2 + 3\pi/2n]\}$. Hence the imaginary axis is the bisector of the former. For this reason, each rotation of the imaginary axis about a multiple of $2\pi/n$ is the line of symmetry of D . Moreover,

$$\{\zeta = it, t \in R\} \cdot \varepsilon^{1/2} = \{\zeta = it, t \in R\} \cdot \varepsilon^{(n+1)/2}.$$

Hence our claim is valid also for all odd j , $j = 0, 1, \dots, 4n - 1$.

If $n = 4k + 3$, $k \geq 0$, then the bisector of $\Lambda_{(n-3)/4}$ divides this sector into two subsectors: $\{w : \arg w \in [\pi/2 - 3\pi/2n, \pi/2 - \pi/2n]\}$ and $\{w : \arg w \in [\pi/2 - \pi/2n, \pi/2 + \pi/2n]\}$. The imaginary axis is the bisector of the latter. Similar argument to the one for $n = 4k + 1$ completes the proof for this choice of n . \square

Theorem 3 *The sets $K_{\mathcal{F}(n)}$ and $L_{\mathcal{F}(n)}$, for positive odd integers n , are $2n$ -fold symmetric.*

Proof Let D be one of the two sets: $K_{\mathcal{F}(n)}$ or $L_{\mathcal{F}(n)}$. Let $w_0 = |w_0|e^{i\varphi_0}$ be an arbitrary point belonging to the boundary of D such that $\arg \varphi_0 \in [0, \pi/2n]$.

It is sufficient to apply Lemma 1. Firstly, the symmetric point to w_0 with respect to the straight line $\varepsilon^{1/4} \cdot \{\zeta = t, t \in R\}$ is $w_1 = |w_0|e^{i(\pi/n - \varphi_0)}$. Secondly, the symmetric point to w_0 with respect to the real axis is $w_2 = |w_0|e^{-i\varphi_0}$. Both points w_1, w_2 also belong to the boundary of D . Consequently,

$$w_1 = \varepsilon^{1/2} \cdot w_2,$$

which results in

$$D = \varepsilon^{1/2} \cdot D.$$

\square

Remark 1 On the basis of this lemma, we can observe that in order to find sets K and L , it is enough to determine their boundaries only in the sector of the measure $\pi/2n$.

2 Extremal Polygons and Functions

Let n be a fixed positive odd integer, $n \geq 5$, and let K denote the Koebe set for $\mathcal{F}^{(n)}$.

We denote by \mathcal{W} , the family of n -fold symmetric polygons W convex in the direction of the real axis and which have $2n$ sides and interior angles $\pi + \pi/n$ and $\pi - 3\pi/n$ alternately.

Suppose that $w_* \in \partial K \cap \Lambda^*$, where ∂K stands for the boundary of K . According to Theorem 1, $-\overline{w_*} \in \partial K$. Consider the straight horizontal line containing a segment $I = \{-\lambda \overline{w_*} + (1 - \lambda)w_* : \lambda \in (0, 1)\}$. There are two possibilities: the intersection of this line with K is either I or the empty set. Assume now that $I \subset K$. We shall see that the second case holds only if $\operatorname{Re} w_* = 0$.

Since K is n -fold symmetric, all points $w_* \cdot \varepsilon^j$, $j = 0, 1, \dots, n-1$ belong to ∂K . On the one hand, $w_* \in \Lambda^*$ means that w_* has the greatest imaginary part among points $w_* \cdot \varepsilon^j$. On the other hand, $w_* \in \partial K$ means that there exists $f_* \in \mathcal{F}^{(n)}$ such that $w_* \in \partial f_*(\Delta)$.

From the convexity of f_* in the direction of the real axis, at least one of the two horizontal rays emanating from w_* is disjoint from $f_*(\Delta)$. Since $\{w_* - t : t \geq 0\} \cap K = I$, it is a ray $l = \{w_* + t : t \geq 0\}$ is disjoint from $f_*(\Delta)$. Taking into account the n -fold symmetry of f_* , all rays $l \cdot \varepsilon^j$, $j = 0, 1, \dots, n-1$ are disjoint from $f_*(\Delta)$.

Observe that the point $w_* \cdot \varepsilon^{(n+1)/2}$ has the lowest imaginary part among points $w_* \cdot \varepsilon^j$, $j = 0, 1, \dots, n-1$. Only if $\arg w_* = \pi/2$, this point is one of two points with the same imaginary part. The convexity in the direction of the real axis of f_* implies that one of two horizontal rays emanating from $w_* \cdot \varepsilon^{(n+1)/2}$ is also disjoint from $f_*(\Delta)$. If this ray is of the form $k_1 = \{w_* \cdot \varepsilon^{(n+1)/2} + t : t \geq 0\}$, then $(k_1 \cdot \varepsilon^j) \cap f_*(\Delta) = \emptyset$ and, consequently, $f_*(\Delta)$ is included in a polygon of the family \mathcal{W} . Indeed, the rays l and $k_1 \cdot \varepsilon^{(n-1)/2}$ form a sector with the vertex in w_* and the opening angle $\pi + \pi/n$. From the n -fold symmetry of f_* , we obtain the polygon mentioned above. The conjugate angle to this opening angle is the vertex angle of the polygon at w_* . It is easy to check that the angle of the polygon between $l \cdot \varepsilon$ and $k_1 \cdot \varepsilon^{(n+1)/2}$ has the measure $\pi - 3\pi/n$.

But there is another possibility, i.e., $k_2 \cap f_*(\Delta) = \emptyset$, where $k_2 = \{w_* \cdot \varepsilon^{(n+1)/2} - t : t \geq 0\}$.

If $\arg w_* = \pi/2$, the set $\mathbb{C} \setminus \{k_2 \cdot \varepsilon^j, j = 0, 1, \dots, n-1\}$ consists of two parts: an unbounded part and a bounded one which is a regular n -gon, see Fig. 1. A regular polygon is convex and it can be treated as the generalization of a set of the family \mathcal{W} . Every second side of this generalized polygon has the length 0.

If $\arg w_* \in [\pi/2 - \pi/n, \pi/2]$, then the set $\mathbb{C} \setminus \{l \cdot \varepsilon^j, k_2 \cdot \varepsilon^j, j = 0, 1, \dots, n-1\}$ is not bounded and it is not convex in the direction of the real axis. Since $w_* \cdot \varepsilon \in \partial f_*(\Delta)$, one of two horizontal rays emanating from this point is also disjoint from $f_*(\Delta)$. If $m_1 = \{w_* \cdot \varepsilon + t : t \geq 0\}$ has no common points with $f_*(\Delta)$, then $w_* \notin \partial f_*(\Delta)$, because $\operatorname{Im}(w_* \cdot \varepsilon) < \operatorname{Im} w_*$ and $\operatorname{Re}(w_* \cdot \varepsilon) < \operatorname{Re} w_*$, a contradiction. For this reason, $m_2 \cap f_*(\Delta) = \emptyset$, where $m_2 = \{w_* \cdot \varepsilon - t : t \geq 0\}$. Hence $(m_2 \cdot \varepsilon^j) \cap f_*(\Delta) = \emptyset$.

In this way, we obtain $3n$ rays emanating from n points: $w_* \cdot \varepsilon^j$, $j = 0, 1, \dots, n-1$. Let us take three rays starting from w_* . These rays are: $l = \{w_* + t : t \geq 0\}$, $k_2 \varepsilon^{-(n+1)/2} = \{w_* - t \cdot \varepsilon^{-(n+1)/2} : t \geq 0\}$ and $m_2 \varepsilon^{-1} = \{w_* - t \cdot \varepsilon^{-1} : t \geq 0\}$. The angles between them and the positive real half-axis are equal to: $0, -\pi/n, \pi - 2\pi/n$.

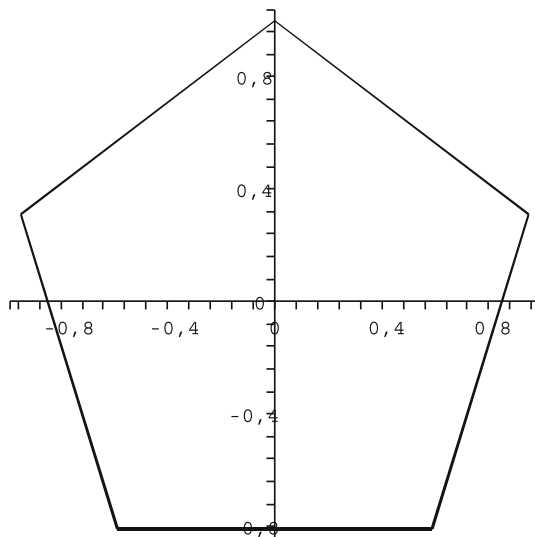


Fig. 1 Extremal n -gon for $n = 5$

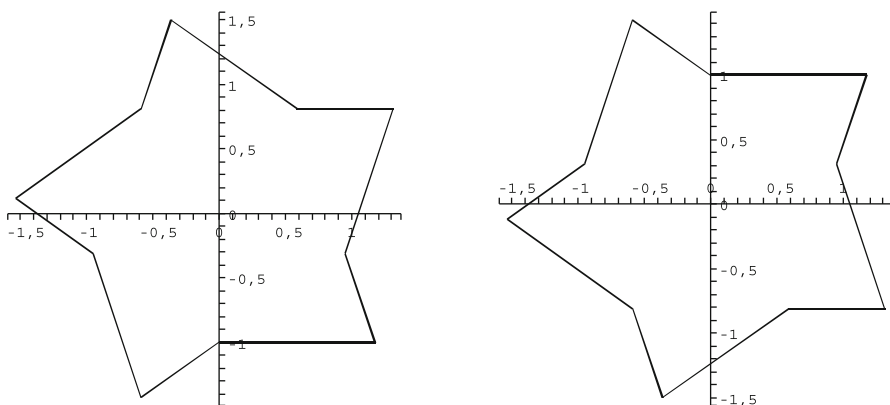


Fig. 2 Extremal polygons for $n = 5$

It means that l lies in the sector with the vertex in w_* and with the sides $k_2\varepsilon^{-(n+1)/2}$ and $m_2\varepsilon^{-1}$. The opening angle of this sector is equal to $\pi - 3\pi/n$.

Consequently, $f_*(\Delta)$ is included in a polygon generated by $k_2\varepsilon^j$ and $m_2\varepsilon^j$, $j = 0, 1, \dots, n-1$. This polygon belongs to the family \mathcal{W} . Two examples of members of \mathcal{W} are shown in Fig. 2.

From now on, we assume that all members of \mathcal{W} are open sets.

Let

$$f_\alpha(z) = \int_0^z (1 - \zeta^n e^{-i\alpha})^{\frac{1}{n}} (1 + \zeta^n e^{-i\alpha/3})^{-\frac{3}{n}} d\zeta, \quad \alpha \in \left[-\frac{3\pi}{n}, \frac{3\pi}{n}\right], \quad (4)$$

$$g_\alpha(z) = \int_0^z (1 - \zeta^n e^{-i\alpha})^{\frac{1}{n}} (1 - \zeta^n e^{-i(\alpha/3+5\pi/3)})^{-\frac{3}{n}} d\zeta, \quad \alpha \in \left[-\frac{2\pi}{n}, \frac{4\pi}{n}\right]. \quad (5)$$

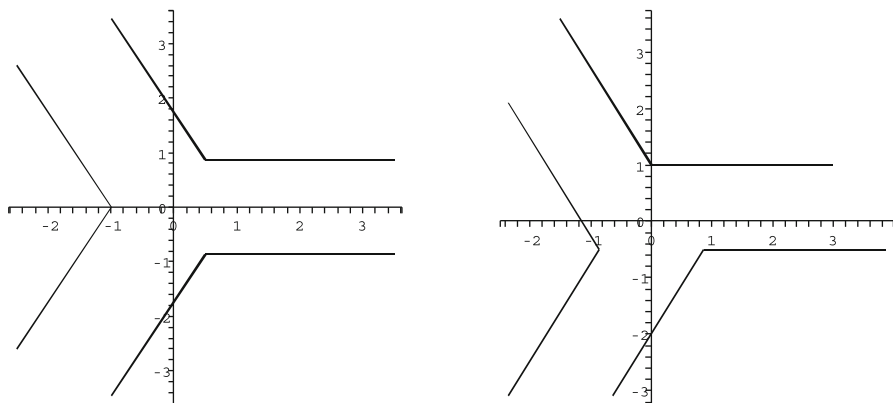


Fig. 3 Extremal polygons for $n = 3$

We choose the principal branch of n -th root. Since the exponential function is periodic, in the above definitions we restrict the range of variability of α to the intervals of length $6\pi/n$. The choice of these intervals depends on the properties of f_α and g_α . Some additional information will be given in Remark 2.

The definition of the family \mathcal{W} may be extended for $n = 3$. In this case, the sets belonging to \mathcal{W} may be treated as generalized polygons. The measure of the angles is equal to $4\pi/3$ and 0 alternately. These sets have the shape of an unbounded three-pointed star, see Fig. 3. Moreover, for $n = 3$, the functions g_α map Δ onto these generalized polygons.

Lemma 2 All functions f_α , $\alpha \in [-\frac{3\pi}{n}, \frac{3\pi}{n}]$, belong to $\mathcal{F}^{(n)}$ for $n = 4k + 1$, $k \geq 1$.

Lemma 3 All functions g_α , $\alpha \in [-\frac{2\pi}{n}, \frac{4\pi}{n}]$, belong to $\mathcal{F}^{(n)}$ for $n = 4k + 3$, $k \geq 0$.

Proof of Lemma 2 At the beginning, we shall show that the functions f_α , $\alpha \in [-3\pi/n, 3\pi/n]$ are univalent. Observe that

$$f'_\alpha(z) = p(z) \cdot \frac{h(z)}{z}, \quad (6)$$

where

$$p(z) = \left(\frac{1 - z^n e^{-ia}}{1 + z^n e^{-ib}} \right)^{1/n}$$

and

$$h(z) = \frac{z}{(1 + z^n e^{-ib})^{2/n}}$$

with $a = n\alpha$, $b = n\alpha/3$.

A Möbius function $p_1(z) = (1 - ze^{-ia})/(1 + ze^{-ib})$, $a, b \in \mathbb{R}$ satisfies the condition $\operatorname{Re} e^{i\beta} p_1(z) > 0$ with some $\beta \in \mathbb{R}$; hence, for all $n \in \mathbb{N}$, the inequality

$\operatorname{Re} e^{i\beta} p(z) = \operatorname{Re} e^{i\beta} p_1(z^n)^{1/n} > 0$ holds with the same β . A function $h_1(z) = z/(1 + ze^{-ib})^2$, $b \in \mathbb{R}$ is starlike, so is $h(z) = \sqrt[n]{h_1(z^n)}$. Combining these two facts with (6), we conclude that

$$\operatorname{Re} e^{i\beta} p(z) = \operatorname{Re} e^{i\beta} \frac{zf'_\alpha(z)}{h(z)} > 0,$$

which means that f_α is close-to-convex and consequently univalent.

Next, we claim that each polygon $f_\alpha(\Delta)$ is a set which is convex in the direction of the real axis. It is sufficient to discuss the argument of the tangent line to $\partial f_\alpha(\Delta)$. Observe that

$$\begin{aligned} \arg \left(\frac{\partial}{\partial \varphi} (f_\alpha(e^{i\varphi})) \right) &= \arg (f'_\alpha(e^{i\varphi}) i e^{i\varphi}) \\ &= \frac{1}{n} \arg (1 + e^{i(\pi + n\varphi - n\alpha)}) - \frac{3}{n} \arg (1 + e^{i(n\varphi - n\alpha/3)}) + \frac{\pi}{2} + \varphi. \end{aligned} \quad (7)$$

Let $\varphi \in [\alpha/3 - \pi/n, \alpha)$. Then, $\pi + n\varphi - n\alpha$ as well as $n\varphi - n\alpha/3$ are in $[-\pi, \pi)$ and from (7) we get

$$\arg \left(\frac{\partial}{\partial \varphi} (f_\alpha(e^{i\varphi})) \right) = \frac{1}{2n} (\pi + n\varphi - n\alpha) - \frac{3}{2n} (n\varphi - n\frac{\alpha}{3}) + \frac{\pi}{2} + \varphi = \frac{\pi}{2} + \frac{\pi}{2n}. \quad (8)$$

The above means that the tangent for φ in $(\alpha/3 - \pi/n, \alpha)$ has the constant argument $\pi/2 + \pi/2n$. Since $f_\alpha(\Delta)$ is a polygon with angles measuring $\pi + \pi/n$ and $\pi - 3\pi/n$ alternately, the argument of the tangent line takes values $\pi/2 + \pi/2n + 2j\pi/n$ and $\pi/2 - \pi/2n + 2j\pi/n$, $j = 1, 2, \dots, n$ alternately. What is more, putting $j = k$ in $\pi/2 + \pi/2n + 2j\pi/n$, we obtain the argument equal to π and putting $j = 3k + 1$ in $\pi/2 - \pi/2n + 2j\pi/n$, we obtain the argument equal to 2π . Hence two of the sides of $f_\alpha(\Delta)$ are horizontal; consequently $f_\alpha \in \mathcal{F}^{(n)}$. \square

The proof of Lemma 3 is similar.

Remark 2 A polygon $f_\alpha(\Delta)$ has vertices in points $f_\alpha(e^{i\alpha}) \cdot \varepsilon^j$ and $f_\alpha(e^{i(\alpha/3 + \pi/n)}) \cdot \varepsilon^j$, $j = 0, 1, \dots, n-1$. These vertices correspond to angles measuring $\pi + \pi/n$ and $\pi - 3\pi/n$, respectively. It is worth pointing out some particular cases of polygons belonging to \mathcal{W} . For $\alpha = -3\pi/2n$ and $\alpha = 3\pi/2n$, they become regular n -gons and for $\alpha = -3\pi/n$, $\alpha = 0$, and $\alpha = 3\pi/n$ these sets are n -pointed stars symmetric with respect to the real axes. The functions $f_{-3\pi/n}$, f_0 , and $f_{3\pi/n}$ have real coefficients. In all other cases, coefficients are nonreal. These particular functions are as follows:

$$\begin{aligned} f_{-3\pi/2n}(z) &= \int_0^z (1 + i\zeta^n)^{-\frac{2}{n}} d\zeta, \quad f_{3\pi/2n}(z) = \int_0^z (1 - i\zeta^n)^{-\frac{2}{n}} d\zeta, \\ f_0(z) &= \int_0^z (1 - \zeta^n)^{\frac{1}{n}} (1 + \zeta^n)^{-\frac{3}{n}} d\zeta, \end{aligned}$$

$$f_{-3\pi/n}(z) = f_{3\pi/n}(z) = \int_0^z (1 + \zeta^n)^{\frac{1}{n}} (1 - \zeta^n)^{-\frac{3}{n}} d\zeta.$$

Likewise, for $\alpha = -\pi/2n$ and $\alpha = 5\pi/2n$ the sets $g_\alpha(\Delta)$ are regular n -gons, and for $\alpha = -2\pi/n$, $\alpha = \pi/n$ and $\alpha = 4\pi/n$ these sets are n -pointed stars. These functions g_α which map Δ on n -pointed stars have real coefficients.

Let $n = 4k + 1$, $k \geq 1$ be fixed. Let us denote by W_α , a set $f_\alpha(\Delta)$ for a fixed $\alpha \in [-3\pi/n, 3\pi/n]$.

Observe that the following equalities hold:

$$f'_{-\alpha}(z) = \overline{f'_\alpha(\bar{z})} \quad \text{for } \alpha \in \left[-\frac{3\pi}{n}, \frac{3\pi}{n}\right]$$

and

$$f'_{\frac{3\pi}{2n}-\gamma}(-z) = \overline{f'_{\frac{3\pi}{2n}+\gamma}(\bar{z})} \quad \text{for } \gamma \in \left[0, \frac{3\pi}{2n}\right],$$

which means that

$$f_{-\alpha}(z) = \overline{f_\alpha(\bar{z})} \quad \text{for } \alpha \in \left[-\frac{3\pi}{n}, \frac{3\pi}{n}\right]$$

and

$$-f_{\frac{3\pi}{2n}-\gamma}(-z) = \overline{f_{\frac{3\pi}{2n}+\gamma}(\bar{z})} \quad \text{for } \gamma \in \left[0, \frac{3\pi}{2n}\right].$$

Consequently,

$$W_{-\alpha} = \overline{W_\alpha} \quad \text{and} \quad -W_{\frac{3\pi}{2n}-\gamma} = \overline{W_{\frac{3\pi}{2n}+\gamma}}.$$

For this reason, a polygon W as well as \overline{W} and $-\overline{W}$ belong to the family \mathcal{W} . From the geometric construction of polygons in \mathcal{W} , it follows that the ratio of lengths of any two adjacent sides of a polygon varies from 0 to infinity as α is changing in $[-3\pi/n, 3\pi/n]$; in each case a polygon is convex in the direction of the real axis. Multiplying sets W_α , $\alpha \in [-3\pi/n, 3\pi/n]$ by $\lambda > 0$ we obtain all members of the set \mathcal{W} .

We have proved one part of the following lemma (the second one can be proved analogously)

Lemma 4

1. $\mathcal{W} = \{\lambda \cdot f_\alpha(\Delta), \lambda > 0, \alpha \in [-3\pi/n, 3\pi/n]\}$ for $n = 4k + 1, k \geq 1$,
2. $\mathcal{W} = \{\lambda \cdot g_\alpha(\Delta), \lambda > 0, \alpha \in [-2\pi/n, 4\pi/n]\}$ for $n = 4k + 3, k \geq 0$.

3 Koebe Sets for $\mathcal{F}^{(n)}$

Let us define

$$F(\alpha) \equiv f_\alpha(e^{i\alpha}), \quad \alpha \in \left[-\frac{3\pi}{n}, \frac{3\pi}{n}\right]$$

and

$$G(\alpha) \equiv g_\alpha(e^{i\alpha}), \quad \alpha \in \left[-\frac{2\pi}{n}, \frac{4\pi}{n}\right].$$

From (4) and (5)

$$F(\alpha) = e^{i\alpha} \int_0^1 (1-t^n)^{\frac{1}{n}} (1+t^n e^{2in\alpha/3})^{-\frac{3}{n}} dt$$

and

$$G(\alpha) = e^{i\alpha} \int_0^1 (1-t^n)^{\frac{1}{n}} (1-t^n e^{i(2n\alpha/3-5\pi/3)})^{-\frac{3}{n}} dt.$$

It can be easily checked that

$$\arg F(\alpha) = \alpha \quad \text{for } \alpha \in \left\{-\frac{3\pi}{n}, -\frac{3\pi}{2n}, 0, \frac{3\pi}{2n}, \frac{3\pi}{n}\right\}, \quad (9)$$

$$\arg G(\alpha) = \alpha \quad \text{for } \alpha \in \left\{-\frac{2\pi}{n}, -\frac{\pi}{2n}, \frac{\pi}{n}, \frac{5\pi}{2n}, \frac{4\pi}{n}\right\}. \quad (10)$$

Theorem 4 *The Koebe set $K_{\mathcal{F}^{(n)}}$, for a fixed $n = 4k + 1$, $k \in \mathbb{N}$, is a bounded and $2n$ -fold symmetric domain such that*

$$\partial K_{\mathcal{F}^{(n)}} \cap \left\{w : \arg w \in \left[-\frac{\pi}{2n}, \frac{\pi}{2n}\right]\right\} = F([- \alpha_F, \alpha_F]), \quad (11)$$

where α_F is the only solution of the equation

$$\arg F(\alpha) = \frac{\pi}{2n} \quad (12)$$

in $[0, 3\pi/2n]$.

Proof Let K denote the Koebe set for $\mathcal{F}^{(n)}$.

Let us consider a polygon $V_\alpha = f_\alpha(\Delta)$ belonging to \mathcal{W} , such that one of its vertices, let say v_* , lies in Λ^* (its argument is in $[\pi/2 - \pi/n, \pi/2]$) and the interior angle at v_* has the measure $\pi(1 + 1/n)$. Suppose additionally that w_* is a point of the boundary of K , such that $\arg w_* = \arg v_*$ and $|w_*| < |v_*|$. We denote the quotient $w_*/v_* = |w_*|/|v_*|$ by λ . Hence $\lambda < 1$.

Since $w_* \in \partial K$, there exists $f_* \in \mathcal{F}^{(n)}$ such that

$$f_*(\Delta) \subset \lambda V_\alpha \subsetneq V_\alpha = f_\alpha(\Delta). \quad (13)$$

Therefore, $f_* \prec f_\alpha$ and $1 = f'_*(0) \leq f'_\alpha(0) = 1$. Consequently $f_* = f_\alpha$, which contradicts (13). It means that $v_* = w_*$, or in other words, w_* coincides with some vertex of $f_\alpha(\Delta)$. Hence w_* is equal to $f_\alpha(e^{i\alpha})$ rotated about a multiple of $2\pi/n$, namely about $2\pi/n \cdot (n-1)/4$. However, it is true only for those α , for which $w_* = F(\alpha) \cdot \varepsilon^{(n-1)/4}$ is in Λ^* .

Observe that for $\alpha \in [-3\pi/n, 3\pi/n]$, we have

$$F(-\alpha) = \overline{F(\alpha)},$$

that is,

$$\arg F(-\alpha) = -\arg F(\alpha).$$

From this and (9), (12) it follows that

$$\left\{ F(\alpha) \cdot \varepsilon^{(n-1)/4} : \alpha \in [-\alpha_F, \alpha_F] \right\}$$

is the boundary of the Koebe set for $\mathcal{F}^{(n)}$ in Λ^* . Combining this with Theorem 3, the equality (11) follows.

Finally, we claim that α_F is the only solution of (12) in $[0, 3\pi/2n]$. On the contrary, assume that there exist two different numbers $\alpha_1, \alpha_2 \in [0, 3\pi/2n]$ such that

$$\arg F(\alpha_1) = \arg F(\alpha_2),$$

or equivalently,

$$\arg f_{\alpha_1}(e^{i\alpha_1}) = \arg f_{\alpha_2}(e^{i\alpha_2}).$$

The sets $W_{\alpha_1} = f_{\alpha_1}(\Delta)$, $W_{\alpha_2} = f_{\alpha_2}(\Delta)$ are polygons of the family \mathcal{W} . This and the definition of \mathcal{W} (or Lemma 4) result in

$$W_{\alpha_1} \subset W_{\alpha_2} \quad \text{or} \quad W_{\alpha_2} \subset W_{\alpha_1}.$$

The normalization of f_{α_1} and f_{α_2} leads to $W_{\alpha_1} = W_{\alpha_2}$. Hence $\alpha_1 = \alpha_2$, a contradiction. This means that (12) has only one solution in the set $[0, 3\pi/2n]$. \square

The above proof gives more. Namely, F is starlike for $\alpha \in [-3\pi/2n, 3\pi/2n]$. Moreover,

$$\arg F\left(\alpha + \frac{3\pi}{n}\right) = \arg F(\alpha) + \frac{3\pi}{n}.$$

It implies that F is starlike for $\alpha \in [-\pi, \pi]$.

Furthermore, it is not difficult to see that there do not exist two different points in the set $\partial K_{\mathcal{F}^{(n)}} \cap \Lambda^*$ with the same imaginary part. For contrary suppose that it is not the case, i.e., there exist v_1 and v_2 such that $v_1 \neq v_2$, $v_1, v_2 \in \partial K_{\mathcal{F}^{(n)}} \cap \Lambda^*$ and $\operatorname{Im} v_1 = \operatorname{Im} v_2$.

With use of an argument similar to those in the proof of previous theorem, we can see that there exist two polygons $V_1, V_2 \in \mathcal{W}$ with vertices v_1, v_2 , respectively. The angles at these vertices have the same measure. Hence the sides of these polygons are pairwise parallel and if $\operatorname{Re} v_1 < \operatorname{Re} v_2$, then $V_1 \subset V_2$. This means that there exist $f_1, f_2 \in \mathcal{F}^{(n)}$ that $f_1(\Delta) = V_1$, $f_2(\Delta) = V_2$, and $f_1 \prec f_2$. But the normalization of f_1 and f_2 is the same, hence $f_1 = f_2$; a contradiction.

Theorem 5 *The Koebe set $K_{\mathcal{F}^{(n)}}$, for a fixed $n = 4k + 3$, $k \geq 0$ is a bounded and $2n$ -fold symmetric domain such that*

$$\partial K_{\mathcal{F}^{(n)}} \cap \left\{ w : \arg w \in \left[\frac{\pi}{2n}, \frac{3\pi}{2n} \right] \right\} = G \left(\left[\alpha_G, \frac{2\pi}{n} - \alpha_G \right] \right), \quad (14)$$

where α_G is the only solution of the equation

$$\arg G(\alpha) = \frac{\pi}{2n} \quad (15)$$

in $[-\pi/2n, \pi/n]$.

Proof A consideration similar to the above shows that α_G is the only solution of the Eq. (15) in $[-\pi/2n, \pi/n]$.

Suppose that $w_* \in \partial K \cap \Lambda^*$. The analogous argument to this in the proof of Theorem 4 yields that K is contained in some polygon W of the family \mathcal{W} .

Let g_* be a function from $\mathcal{F}^{(n)}$ for which $w_* \in \partial g_*(\Delta)$. We have $g_*(\Delta) \subset W = g_\alpha(\Delta)$ for some $\alpha \in [-\pi/2n, 5\pi/n]$. For this reason $g_* \prec g_\alpha$, but taking into account the normalization of both functions, we obtain $g_* = g_\alpha$. Hence $w_* = g_\alpha(e^{i\alpha}) \cdot \varepsilon^{(n-3)/4}$, but only if $w_* \in \Lambda^*$.

For $\alpha \in [-\pi/2n, \pi/n]$,

$$G\left(\frac{2\pi}{n} - \alpha\right) = \varepsilon \overline{G(\alpha)},$$

and so

$$\arg G\left(\frac{2\pi}{n} - \alpha\right) = \frac{2\pi}{n} - \arg G(\alpha).$$

From this, (10) and (15), we conclude that

$$\left\{ G(\alpha) \cdot \varepsilon^{(n-3)/4} : \alpha \in \left[\alpha_G, \frac{2\pi}{n} - \alpha_G \right] \right\}$$

is the boundary of the Koebe set for $\mathcal{F}^{(n)}$ in Λ^* . Theorem 3 concludes the proof of our theorem. \square

Now, we can derive the Koebe constant for $\mathcal{F}^{(n)}$.

Theorem 6 *For a fixed positive odd integer n , $n \geq 3$ and for every function $f \in \mathcal{F}^{(n)}$ the disk Δ_{r_n} , where $r_n = B(1/n, 1/2n + 1/2)/n\sqrt[n]{4}$, is included in $f(\Delta)$. The number r_n cannot be increased.*

The symbol B stands for the Beta and Δ_r , $r > 0$ means $\Delta_r = \{\zeta \in \mathbb{C} : |\zeta| < r\}$.

Proof According to Theorems 4 and 5, the Koebe constant is equal to

$$\min \{|F(\alpha)| : \alpha \in [-\alpha_F, \alpha_F]\} \quad \text{for } n = 4k + 1,$$

or

$$\min \left\{ |G(\alpha)| : \alpha \in \left[\alpha_G, \frac{2\pi}{n} - \alpha_G \right] \right\} \quad \text{for } n = 4k + 3.$$

But

$$|F(\alpha)|^2 \geq \left(\int_0^1 (1-t^n)^{\frac{1}{n}} \operatorname{Re}(1+t^n e^{2in\alpha/3})^{-\frac{3}{n}} dt \right)^2$$

and the integrand in this expression is nonnegative; thus

$$|F(\alpha)| \geq \int_0^1 (1-t^n)^{\frac{1}{n}} \operatorname{Re} q_F(\alpha, t) dt, \quad q_F(\alpha, t) = (1+t^n e^{2in\alpha/3})^{-\frac{3}{n}}.$$

Likewise,

$$|G(\alpha)| \geq \int_0^1 (1-t^n)^{\frac{1}{n}} \operatorname{Re} q_G(\alpha, t) dt, \quad q_G(\alpha, t) = (1-t^n e^{i(2n\alpha/3-5\pi/3)})^{-\frac{3}{n}}.$$

It is easy to check that for $n \geq 3$ the functions $p(z) = (1 \pm t^n z)^{-3/n}$ are convex in Δ and they have real coefficients. This means that

$$\operatorname{Re}(1 \pm t^n z)^{-\frac{3}{n}} \geq (1+t^n)^{-\frac{3}{n}}. \quad (16)$$

Applying (16) for both q_F and q_G , we get

$$|F(\alpha)| \geq q_0 \quad \text{and} \quad |G(\alpha)| \geq q_0,$$

where

$$q_0 = \int_0^1 (1-t^n)^{\frac{1}{n}} (1+t^n)^{-\frac{3}{n}} dt. \quad (17)$$

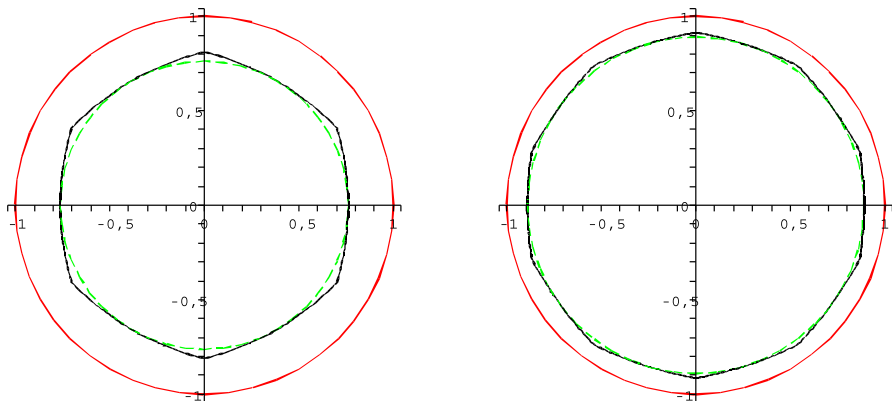


Fig. 4 Koebe domain (solid line) and Koebe disk (dashed line) for $\mathcal{F}^{(3)}$ and $\mathcal{F}^{(5)}$

This results in

$$\min \{|F(\alpha)| : \alpha \in [-\alpha_F, \alpha_F]\} = |F(0)|$$

and

$$\min \left\{ |G(\alpha)| : \alpha \in \left[\alpha_G, \frac{2\pi}{n} - \alpha_G \right] \right\} = \left| G\left(\frac{\pi}{n}\right) \right|.$$

Moreover, substituting $t^n = \tan^2(x/2)$ in (17), we get

$$q_0 = \frac{2}{n\sqrt[n]{4}} \int_0^{\pi/2} (\sin x)^{\frac{2}{n}-1} (\cos x)^{\frac{1}{n}} dx = \frac{1}{n\sqrt[n]{4}} B\left(\frac{1}{n}, \frac{1}{2n} + \frac{1}{2}\right).$$

□

Corollary 1 For a fixed positive odd integer n , $n \geq 3$, the Koebe constant for $\mathcal{F}^{(n)}$ is equal to $r_n = B(1/n, 1/2n + 1/2)/n\sqrt[n]{4}$.

The Koebe sets and the Koebe disks for $n = 3$ and $n = 5$ are shown in Fig. 4.

Remark 3 The results established in Theorem 4 and in Corollary 1 are actually valid also for $n = 1$. They were obtained by Złotkiewicz and Reade in [6].

One can check that for $n = 1$, the function F takes the form

$$F(\alpha) = e^{i\alpha} \int_0^1 \frac{1-t}{(1+te^{2i\alpha/3})^3} dt = \frac{e^{2i\alpha/3}}{4 \cos(\alpha/3)}.$$

The Eq. (12) gives $\alpha_F = 3\pi/4$; thus the boundary of the Koebe set in the upper half-plane can be written as follows

$$u = \frac{\cos(2\alpha/3)}{4 \cos(\alpha/3)}, \quad v = \frac{1}{2} \sin(\alpha/3), \quad \alpha \in \left[-\frac{3\pi}{4}, \frac{3\pi}{4} \right].$$

This fact, we can rewrite in a different way

$$K_{\mathcal{F}} = \{w \in \mathbb{C} : 8|w| (|w| + |\operatorname{Re} w|) < 1\}.$$

The extremal functions f_{α} given by (4) are of the form

$$f_{\alpha}(z) = \frac{z + Bz^2}{(1 + ze^{-i\alpha/3})^2}, \quad B = i \sin(\alpha/3)e^{-2i\alpha/3}$$

and

$$g_{\alpha}(z) = -f_{\alpha}(-z),$$

where $\alpha \in [-3\pi/4, 3\pi/4]$. The image set $f_{\alpha}(\Delta)$ for a fixed $\alpha \in (-3\pi/4, 3\pi/4)$ coincides with the plane with a horizontal ray excluded. For $\alpha = -3\pi/4, 3\pi/4$, the sets $f_{\alpha}(\Delta)$ are half-planes.

Moreover, $r_1 = B(1, 1)/4 = 1/4$.

4 Covering Domains for $\mathcal{F}^{(n)}$

Theorem 7 *The covering set $L_{\mathcal{F}^{(n)}}$ for odd $n \geq 5$ is a bounded and $2n$ -fold symmetric domain such that*

$$\partial L_{\mathcal{F}^{(n)}} \cap \left\{ w : \arg w \in \left[0, \frac{\pi}{2n}\right] \right\} = H \left(\left[0, \frac{\pi}{2n}\right] \right). \quad (18)$$

In the proof of this theorem, we need the following lemma.

Lemma 5 *Let $n \geq 5$ be a fixed odd integer. If $f \in \mathcal{F}^{(n)}$ and $w \in f(\Delta) \cap \Lambda^*$, then $f(\Delta)$ contains a polygon $W \in \mathcal{W}$ such that W has one of its vertices at w and the interior angle at w has the measure $\pi - 3\pi/n$.*

Proof Let $f \in \mathcal{F}^{(n)}$ and $w \in f(\Delta) \cap \Lambda^*$, i.e., $\arg w \in [\pi/2 - \pi/n, \pi/2]$. Because of the n -fold symmetry of f every point $w \cdot \varepsilon^j$, $j = 0, 1, \dots, n-1$ belongs to $f(\Delta)$.

It can be easily checked that

$$\max \left\{ \operatorname{Im} (w \cdot \varepsilon^j), j = 0, 1, \dots, n-1 \right\} = \operatorname{Im} (w)$$

and

$$\min \left\{ \operatorname{Im} (w \cdot \varepsilon^j), j = 0, 1, \dots, n-1 \right\} = \operatorname{Im} (w \cdot \varepsilon^{2k+1}).$$

Let $w_1 = w \cdot \varepsilon$ and $w_2 = w \cdot \varepsilon^{2k}$. The point w_1 has the second biggest imaginary part among points $w, w \cdot \varepsilon, \dots, w \cdot \varepsilon^{n-1}$. Likewise, w_2 has the second lowest imaginary part among those points.

Let, moreover, l_1 and l_2 stand for two horizontal rays emanating from w_1 and w_2 : $l_1 = \{w_1 + t : t \geq 0\}$, $l_2 = \{w_2 + t : t \geq 0\}$, respectively.

From the inequality $\operatorname{Im} w_2 > \operatorname{Im} w \cdot \varepsilon^{2k+1}$, we conclude that the point $w \cdot \varepsilon^{2k+1}$ lies on the opposite side of the straight line which contains l_2 with respect to the origin. As a consequence, w_1 lies on the other side of the straight line including $l_2 \cdot \varepsilon^{-2k}$ with respect to the origin. Hence, two rays l_1 and $l_2 \cdot \varepsilon^{-2k}$ have a common point, let say w_0 .

We shall show that w_0 also belongs to $f(\Delta)$. Suppose, contrary to our claim, that $w_0 \notin f(\Delta)$. The points w_0, w_1 lie on the ray l_1 and $w_1 \in f(\Delta)$. Therefore, taking into account the convexity in the direction of the real axis of f , a ray $m_1 = \{w_0 + t : t \geq 0\}$ is disjoint from $f(\Delta)$.

Since w_0, w belong to $l_2 \cdot \varepsilon^{-2k}$, the points $w_0 \cdot \varepsilon^{2k}, w \cdot \varepsilon^{2k}$ belong to l_2 . Moreover, $w_0 \cdot \varepsilon^{2k} \notin f(\Delta)$ and $w_2 \in f(\Delta)$. Consequently, $m_2 = \{w_0 \cdot \varepsilon^{2k} + t : t \geq 0\}$ is disjoint from $f(\Delta)$, and, generally, $m_2 \varepsilon^j \cap f(\Delta) = \emptyset$, $j = 0, 1, \dots, n-1$.

We have proved that the rays m_1 and $m_2 \varepsilon^{-2k}$ with the common vertex w_0 are disjoint from $f(\Delta)$. It means that the reflex sector with the vertex in w_0 and these two rays as the sides has no common points with $f(\Delta)$. But w_1 lies in this reflex sector; hence $w_1 \notin f(\Delta)$, a contradiction.

From the argument given above all points $w \varepsilon^j, w_0 \varepsilon^j$, $j = 0, 1, \dots, n-1$ belong to $f(\Delta)$. Applying n -fold symmetry and the convexity of f in the direction of the real axis, we can see that a polygon W with succeeding vertices at points $w, w_0, w \varepsilon, w_0 \varepsilon, \dots, w \varepsilon^{n-1}, w_0 \varepsilon^{n-1}$ is contained in $f(\Delta)$. It is easy to check that this polygon has the interior angles $\pi - 3\pi/n$ and $\pi + \pi/n$ alternately. For this reason, W is in \mathcal{W} . \square

According to Lemmas 2 and 3, every function in $\mathcal{F}^{(n)}$ mapping Δ onto a polygon of the family \mathcal{W} has the form (4) and (5) with appropriately taken α . These functions may be written in the form

$$f_\beta(z) = \int_0^z (1 + \zeta^n e^{-3in\beta})^{\frac{1}{n}} (1 - \zeta^n e^{-in\beta})^{-\frac{3}{n}} d\zeta, \quad \beta \in \left[0, \frac{2\pi}{n}\right], \quad (19)$$

$$g_\beta(z) = \int_0^z (1 + \zeta^n e^{-3in\beta})^{\frac{1}{n}} (1 - \zeta^n e^{-in\beta})^{-\frac{3}{n}} d\zeta, \quad \beta \in \left[\frac{\pi}{n}, \frac{3\pi}{n}\right], \quad (20)$$

equivalent to (4) and (5).

In fact, the functions defined by (4) and (19) are connected by the relation $\beta = \alpha/3 + \pi/n$ and the functions in (5) and (20) are connected by $\beta = \alpha/3 + 5\pi/3n$.

Let us define

$$H(\beta) = f_\beta(e^{i\beta}) \quad \text{for } \beta \in \left[0, \frac{2\pi}{n}\right],$$

and

$$H(\beta) = g_\beta(e^{i\beta}) \quad \text{for } \beta \in \left[\frac{\pi}{n}, \frac{3\pi}{n}\right].$$

Hence

$$H(\beta) \equiv e^{i\beta} \int_0^1 (1 + t^n e^{-2in\beta})^{\frac{1}{n}} (1 - t^n)^{-\frac{3}{n}} dt, \quad \beta \in \mathbb{R}.$$

Observe that

$$\arg H(\beta) = \beta \quad \text{for} \quad \beta = \frac{\pi}{2n} \cdot j, \quad j = 0, 1, \dots, 4n - 1. \quad (21)$$

Furthermore,

$$H\left(\beta + \frac{\pi}{n}\right) = e^{i\frac{\pi}{n}} H(\beta). \quad (22)$$

Now, we can prove Theorem 7.

Proof of Theorem 7 Let L denote the covering set for $\mathcal{F}^{(n)}$. We additionally assume that $n = 4k + 1$, $k \geq 1$. The proof for the case $n = 4k + 3$, $k \geq 0$ is almost similar.

Let us consider a polygon $W_\beta = f_\beta(\Delta)$ belonging to \mathcal{W} , such that one of its vertices, let say w^* , lies in Λ^* and the interior angle at w^* has the measure $\pi(1 - 3/n)$. Suppose additionally that v^* is a point of the boundary of L such that $\arg v^* = \arg w^*$ and $|v^*| > |w^*|$. We denote the quotient $v^*/w^* = |v^*|/|w^*|$ by μ . Hence $\mu > 1$.

Since $v^* \in \partial L$, there exists $f^* \in \mathcal{F}^{(n)}$ such that v^* is a boundary point of $f^*(\Delta)$. From Lemma 5

$$f^*(\Delta) \supset \mu W_\beta \supsetneq W_\beta = f_\beta(\Delta). \quad (23)$$

Therefore, $f_\beta < f^*$ and $1 = f'_\beta(0) \leq f^{*'}(0) = 1$. Consequently $f_\beta = f^*$, which contradicts (23). It means that $w^* = v^*$, or in other words, v^* coincides with some vertex of $f_\beta(\Delta)$. Hence v^* is equal to $f_\beta(e^{i\beta})$ rotated about a multiple of $2\pi/n$, namely about $2\pi/n \cdot (n-1)/4$. It is enough to take such β that $v^* = H(\beta) \cdot \varepsilon^{(n-1)/4}$ is in Λ^* . From this, we conclude that $\beta \in [0, \pi/2n]$. \square

Theorem 8 For a fixed odd integer $n \geq 5$ and for every function $f \in \mathcal{F}^{(n)}$, the set $f(\Delta)$ is included in Δ_{R_n} , where $R_n = B(1/n, 1/2 - 3/2n)/n\sqrt[4]{4}$. The number R_n cannot be decreased.

Proof We have

$$|H(\beta)| \leq \int_0^1 \left| (1 + t^n e^{-2in\beta})^{\frac{1}{n}} (1 - t^n)^{-\frac{3}{n}} \right| dt \leq \int_0^1 \frac{(1 + t^n)^{\frac{1}{n}}}{(1 - t^n)^{\frac{3}{n}}} dt = |H(0)|.$$

It can be shown that $H(0) = B(1/n, 1/2 - 3/2n)/n\sqrt[4]{4}$. \square

Corollary 2 For a fixed odd integer $n \geq 5$, the covering constant for $\mathcal{F}^{(n)}$ is equal to $R_n = B(1/n, 1/2 - 3/2n)/n\sqrt[4]{4}$.

The results presented above are valid for positive odd integers greater than or equal to 5. In the last part of this section, we turn to the case $n = 3$.

As it was said in Section 2 (see also Fig. 3) for $n = 3$ and $\beta \in [\pi/3, \pi] \setminus \{\pi/2, 5\pi/6\}$, the functions given by (20) map Δ onto the polygons with the interior angles $4\pi/3$ and

0 alternately, and the vertices in points $a \cdot \varepsilon^j, \infty \cdot a \cdot \varepsilon^j, j = 0, 1, 2$ alternately, where $a = g_\beta(e^{i\beta}) = H(\beta)$. Both sides adjacent to every vertex in infinity are parallel. Hence $g_\beta(\Delta)$ are star-shaped sets with three unbounded strips. The strips have the direction $\pi/3, \pi, 5\pi/3$ if $\beta \in [\pi/3, \pi/2) \cup (5\pi/6, \pi]$ and $0, 2\pi/3, 4\pi/3$ if $\beta \in (\pi/2, 5\pi/6)$. The thickness of the strips is changing as β varies in $\beta \in [\pi/3, \pi] \setminus \{\pi/2, 5\pi/6\}$, but when β tends to $\pi/2$ or $5\pi/6$, the thickness of the strips tends to 0.

For $\beta = \pi/2$ and $\beta = 5\pi/6$, the functions

$$g_{\frac{\pi}{2}}(z) = \int_0^z \frac{1}{(1 - i\zeta^3)^{2/3}} d\zeta$$

and

$$g_{\frac{5\pi}{6}}(z) = \int_0^z \frac{1}{(1 + i\zeta^3)^{2/3}} d\zeta$$

map Δ onto the equilateral triangles symmetric with respect to the imaginary axis. The first triangle has one of its vertices in the point ic , the second one - in the point $-ic$, where

$$c = \frac{B(\frac{1}{3}, \frac{1}{6})}{3\sqrt[3]{4}} = 1.76 \dots$$

Let

$$\Omega_0 = \left\{ w : \operatorname{Re} w \geq 0, |\operatorname{Im} w| < \frac{1}{2}c \right\}.$$

Theorem 9 *The covering domain $L_{\mathcal{F}(3)}$ is an unbounded and 6-fold symmetric domain*

$$L_{\mathcal{F}(3)} = \bigcup_{j=0}^5 e^{j\frac{\pi}{3}i} \cdot \Omega_0.$$

Proof Let L denote the covering set for $\mathcal{F}^{(n)}$ and let L^* stand for $\bigcup_{j=0}^5 e^{j\pi i/3} \cdot \Omega_0$.

At the beginning, we can see that L includes six-pointed star obtained as a union of $g_{\pi/2}(\Delta)$ and $g_{5\pi/6}(\Delta)$. We know that for $\beta \in (\pi/2, 5\pi/6)$ each set $g_\beta(\Delta)$ contains a part of a horizontal strip between two rays emanating from a/ε and a , where $a = H(\beta)$. From (21), it follows that the arguments of these points vary continuously from $-\pi/6$ to $\pi/6$ for the point a/ε and from $\pi/2$ to $5\pi/6$ for the point a . This and the symmetry of L with respect to the imaginary axis result in $L^* \subset L$.

Now, we shall prove that $L \subset L^*$. On the contrary, assume that $w_0 \notin L^*$ but $w_0 \in L$. It means that there exists a function $f_0 \in \mathcal{F}^{(3)}$ such that $w_0 \in f_0(\Delta)$. Without loss of generality, we can assume that $\arg w_0 \in (0, \pi/6)$ because of Lemma 1 and Remark 1.

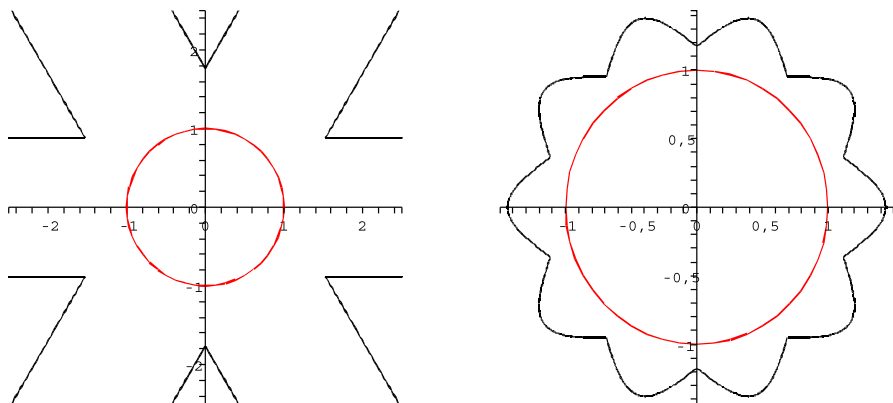


Fig. 5 Covering domains for $\mathcal{F}^{(3)}$ and $\mathcal{F}^{(5)}$

From the 3-fold symmetry of f_0 , we know that $w_0\varepsilon, w_0\varepsilon^2 \in f_0(\Delta)$. Moreover,

$$\operatorname{Im} w_0 = |w_0| \sin \varphi_0 < |w_0| \sin \left(\varphi_0 + \frac{2\pi}{3} \right) = \operatorname{Im}(w_0\varepsilon),$$

because $\varphi_0 = \arg w_0 \in (0, \pi/6)$.

Observe that the point $w_1 = \{w_0 - t : t \geq 0\} \cap (\varepsilon \cdot \{w_0 - t : t \geq 0\})$ also belongs to $f_0(\Delta)$. If it were not the case, the points $w_1\varepsilon, w_1\varepsilon^2$ would not be in $f_0(\Delta)$ either. But $w_1, w_1\varepsilon^2 \in \{w_0 - t : t \geq 0\}$. Combining $w_1, w_1\varepsilon^2 \notin f_0(\Delta)$ with $w_0 \in f_0(\Delta)$ yields that the segment connecting w_1 and $w_1\varepsilon^2$ has no common points with $f_0(\Delta)$. From this and the 3-fold symmetry, all three segments connecting $w_1, w_1\varepsilon, w_1\varepsilon^2$ and, as a consequence, the equilateral triangle T with vertices in these points, would be disjoint with $f_0(\Delta)$, a contradiction. This means that $w_1, w_1\varepsilon, w_1\varepsilon^2 \in f_0(\Delta)$, which results in

$$T \subset f_0(\Delta). \quad (24)$$

But

$$g_{\frac{5\pi}{6}}(\Delta) \subset T \quad \text{and} \quad g_{\frac{5\pi}{6}}(\Delta) \neq T. \quad (25)$$

From (24) and (25), $g_{5\pi/6}$ is subordinated to f , but $g_{5\pi/6}$ and f have the same normalization, a contradiction. It means that if $w_0 \in L$, then $w_0 \in L^*$, which completes the proof. \square

The covering domains for $\mathcal{F}^{(3)}$ and $\mathcal{F}^{(5)}$ are shown in Fig. 5.

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